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FLOW OF A RELAXING GAS AROUND A THIN CONE OF REVOLUTION

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ABSTRACT

The problem of supersonic flow around thin bodies of revolution by a gas, in which chemical reactions and relaxation of internal degrees of freedom take place, is reduced to the integration of a single differential equation. It is assumed that the linearized flow equations hold and that functions of state exist. Using Laplace transformation methods, the author discusses flow close to the initial frozen Mach line, flow close to the surface of the cone, and the region far removed from the axis and the initial frozen Mach line.

The problem of supersonic flow around thin bodies of revolution by a gas, in which chemical reactions and relaxation of internal degrees of freedom take place, can be reduced to the integration of the equation

$$K\,(\lambda_f^2\Phi_{xx} - (1/r)\,(r\Phi_r)_r)_x + \lambda_e^2\Phi_{xx} - (1/r)\,(r\Phi_r)_r = 0, \qquad (1)$$

as can be shown (which is written in a cylindrical coordinate system, the x - axis of which coincides with the axis of the body of revolution and which is parallel to the direction of the undisturbed stream (Figure 1)). In this equation, K is the parameter which is proportional to the length of relaxation in the undisturbed stream $U_\infty\tau_\infty$ (the index ∞ designates the quantity referring to the undisturbed region); τ is the relaxation time; Φ is the potential of the velocity of disturbance, determined by the relationship $\Phi_x = u$, $\Phi_r = v$; u and v are the velocity components of the disturbance along the x and r axes; $\lambda_f = \sqrt{M_f^2 - 1}$, $\lambda_e = \sqrt{M_e^2 - 1}$, M_f and M_e are the Mach numbers which are equal respectively to U_∞/a_f and U_∞/a_ε. In general, in an examination of the propagation of weak disturbances in a relaxing medium two velocities of sound arise: frozen a_f and equilibrium a_e (Ref. 1). Since a_f is always greater than a_e, $M_f < M_e$, and the condition providing for the supersonic nature of the flow will be $M_f > 1$.

The conditions under which equation (1) will hold can be reduced primarily to two assumptions. Under the first assumption, it is assumed that the conditions for the linearization of the equations describing the flow field are fulfilled, i.e. the streamlined body is assumed to be thin. Under the second assumption, it is assumed that the non-equilibrium states which arise allow a thermodynamic description, i.e. a function of state exists - let us call enthalpy h - which is a function not only of density ρ and pressure P, but also a function of the parameter q which characterizes the deviation from complete thermodynamic equilibrium. For example, in concrete cases it is possible to let the oscillation or rotation temperature, degree of dissociation, etc. be such a parameter.

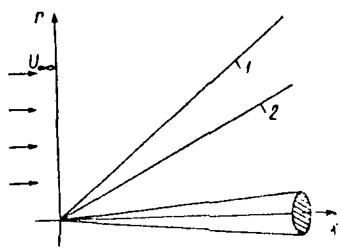

Figure 1

Figure Showing the Flow of a Relaxing Gas Around a Thin Cone of Revolution Under Zero Angle of Attack (1 - Initial Frozen Mach Line; 2 - Initial Equilibrium Mach Line).

Equations which are analogous to (1) were obtained and analyzed in the works (Ref. 2 - Ref. 4). The characteristics of nonequilibrium flow around thin, two-dimensional profiles were studied in detail with their help. However, in practice, flow around three-dimensional bodies is also of great interest; the simplest example of this is a thin cone of revolution.

Before writing out the corresponding boundary conditions for equation (1) in explicit form, let us transform it - introducing dimensionless variables - according to the relationships:

$$\Phi = U_\infty K \varphi,\ x = Kx',\ r = Kr'/\lambda_f,\ a = \lambda_e^2/\lambda_f^2 > 1;$$

then, changing to new independent variables, $\xi = x' - r'$, $\eta = r'$. Thus, instead of (1), we obtain

$$\begin{gathered} 2\varphi_{\xi\xi\eta} - \varphi_{\xi\eta\eta} - [(1/\eta) - 2]\,\varphi_{\xi\eta} + \\ + [(1/\eta) + a - 1]\,\varphi_{\xi\xi} - \varphi_{\eta\eta} - (1/\eta)\,\varphi_\eta + (1/\eta)\,\varphi_\xi = 0. \end{gathered} \tag{2}$$

The boundary conditions for this equation are:

$$\varphi = \begin{cases} 0 & \text{for} \quad \xi < 0 \\ < N & \text{for} \quad \eta \to \infty \quad (N = \text{const}) \end{cases} \tag{3}$$

and

$$\lim_{\eta \to 0} \eta (\varphi_\eta - \varphi_\xi) = \varepsilon^2 \xi. \tag{4}$$

here ε is the tangent of the half angle at the apex of the cone. Equation (4) arises from the requirement, which is customary in the theory of thin bodies, that $\lim_{r \to 0} rv = R \frac{dR}{dx}$, where $R(x)$ is the generatrix of the streamlined, axisymmetric body.

We can solve equation (2) with the aid of the Laplace transformation. Designating the Laplacian form of the function by the same symbol, but with a bar above, we obtain in image space the ordinary differential equation of the second order for: $\bar{f}$ $(f = \sqrt{\eta}\varphi)$:

$$\bar{f}_{\eta\eta} - 2p\bar{f}_\eta - \left[p^2 \left(\frac{a-1}{p+1}\right) - \frac{1}{(2\eta)^2}\right] \bar{f} =$$

$$= -(p+1)^{-1} \left\{ \left[2\frac{d}{d\eta} + (a-1) \right] f_\xi(0, \eta) - \right.$$

$$\left. - \left[\frac{d^2}{d\eta^2} - 2(p+1)\frac{d}{d\eta} + 1/(2\eta)^2 - p(a-1) \right] f(0, \eta) \right\}. \tag{5}$$

Setting the right part of (5) equal to zero, we obtain the equation

$$\bar{f}_{\eta\eta} - 2p\bar{f}_\eta - \left[p^2 \left(\frac{a-1}{p+1}\right) - \frac{1}{(2\eta)^2}\right] \bar{f} = 0. \tag{6}$$

This change is justified if the solution of equation (6), substituted in the right part of (5), makes it zero. Below, we shall verify the fact that the solution which is of interest for us actually satisfies this condition.

Completing the substitution $\bar{f} = e^{p\eta} z$ and $t = p\,\eta\sigma$ $\left\{\sigma^2 = \frac{a+p}{1+p}\right\}$, we change equation (6) to the form

$$z_{tt} - [1 - (2t)^{-2}]\, z = 0, \tag{7}$$

the general solution of which will be

$$z = A(p)\sqrt{t}\, I_0(t) + B(p)\sqrt{t}\, K_0(t). \tag{8}$$

Because of the requirement for the boundedness of the potential for $\eta \to \infty$, we must set $A = 0$. Then, for $\bar{\varphi}$ we obtain

$$\bar{\varphi} = B(p)\sqrt{p\sigma}\, e^{p\eta} K_0(p\eta\sigma), \tag{9}$$

where the constant $B(p)$ must be determined with the aid of the boundary condition (4). As a result, we find that

$$\overline{\varphi}(p,\eta) = -(\varepsilon^2/p^2)\, e^{p\eta} K_0(p\eta\sigma). \tag{10}$$

will be a solution for the problem in image space which satisfies the boundary conditions.

Since it is rather difficult to invert $\bar{\varphi}$ directly with the aid of the inversion theorem, let us study the behavior of the solution in the limiting cases $p \to 0$, $p \to \infty$ and $\eta \to 0$, where the right part of (10) is greatly simplified.

1. Flow Close to the Initial Frozen Mach Line.

According to the transformation, the region $\xi \ll 1$, which corresponds to dimensionless variable distances which are considerably smaller than the length of relaxation close to the initial frozen Mach line, envelops the zone near the frozen flow. From the theory of Laplace transformation, it is known (Ref. 5) that the asymptotic expansion of the image for $p \to \infty$ corresponds to the expansion of the original in a power series around $\xi = 0$. Therefore, utilizing the expansion for $K_0(p\,\eta\sigma)$ for large values of the argument and separating σ and σ^{-1} in series of p^{-1}, we find that for $p \to \infty\, \varphi\,(p, \eta)$ it is possible to represent the following in the form

$$\overline{\varphi}(p,\eta) \sim -\varepsilon^2 \sqrt{\frac{\pi}{2\eta}} \exp\left\{-\left(\frac{a-1}{2}\right)_\eta\right\} \sum_{n=0}^{\infty} \frac{A_n}{p^{n+5/2}}, \tag{11}$$

and the potential itself for $\xi \to 0$

$$\varphi(\xi,\eta) \approx -\varepsilon^2 \sqrt{\frac{\pi}{2\eta}} \exp\left\{-\left(\frac{a-1}{2}\right)_\eta\right\} \sum_{n=0}^{\infty} \left[\frac{A_n}{\Gamma\left(n+\frac{3}{2}\right)}\right] \xi^{n+3/2}. \tag{12}$$

The general form of the coefficients A_n is:

$$A_n = \sum_{-n}^{n} c_i \eta^i, \tag{13}$$

where c_i are the constants depending on i.

First of all, it follows from formula (12) that the transformation from (5) to (6) was justified, since $\varphi(0, \eta) = \varphi'_\xi(0, \eta) = 0$. In the second place, the potential itself and its derivatives - i.e., the velocities of disturbance - decrease exponentially with an increase in η. Therefore, close to the apex of the cone along the frozen Mach line, a disturbance is formed which decreases rapidly with distance from the axis, just as in plane flow. The main increase in the disturbance occurs along the equilibrium Mach line, but - in contrast to the equilibrium flow - the velocities of the disturbance on this line will not be equal to zero.

2. Field of Flow Close to the Surface of the Cone.

A picture of the flow close to the axis is of great practical interest, because the value for the velocity of the disturbance on the cone surface makes it possible to find the pressure coefficient c_p and, consequently, to calculate the resistance experienced by the body. Since the body being examined is thin, we expand the expression for $\bar{u}\ (p,\eta) = -\ (\varepsilon^2/p)\ e^{p\eta}K_0\ (p\ \eta\sigma)$ and $\bar{v}\ (p,\eta) = (\varepsilon^2/p)\ e^{p\eta}\ K_1\ (p\ \eta\sigma)$ in series for $\eta \to 0$, and we limit ourselves to the dominant terms only. We have:

$$\bar{u}\,(p,\ \eta) \approx (\varepsilon^2/p)\,[\ln(p\,\eta\sigma/2) + C],$$
$$\bar{v}(p,\ \eta) \approx \varepsilon^2/p^2\,\eta, \tag{14}$$

where C is the Euler constant. Transforming these expressions and changing to dimensional variables, we find:

$$u \approx -\,\varepsilon^2 U_\infty \left\{ \ln\left(\frac{2x}{\lambda_f\, r}\right) - F\left(\frac{x}{K},\, a\right)\right\}.$$
$$v \approx \varepsilon^2\, x/r. \tag{15}$$

Here $F\ (\xi,\ a) = (1/2)\ [\ln a + \mathrm{Ei}(-\xi) - \mathrm{Ei}(-a\xi)]$. Thus, the pressure coefficient on the surface of a thin cone equals

$$c_p = 2\varepsilon^2 \left\{ \ln(2/\varepsilon\lambda_f) - \frac{1}{2} - F\,(x/K, a)\right\}. \tag{16}$$

Formula (16) differs from the corresponding classical expression by the presence of an additional term $F(x/K,\ a)$. It can be readily seen that, close to the apex of the cone, for $x \to 0$ $F\ (x/K,\ a) \to 0$, C_p agrees with the classical value calculated on the basis of frozen velocity of sound. For $x \to \infty$ $F > (1/2)\ \ln a$, C_p strives to the classical value, but for the equilibrium speed of sound. This means that the entire effect of the relaxation process, which takes place in the medium, is concentrated close to the apex of the cone and at distances which considerably exceed the length of relaxation, and the flow proves to be stable (see Figure 2).

3. The Flow for Large ξ and η.

The region $\xi \to \infty$ corresponds in image space to the region $p \to 0$. However, we wish to study the region which is not only far from the initial frozen Mach line, but also far from the axis. Therefore, it is impossible to regard the product $p\eta$ as small. Since for $p \to 0$ $\sigma \to \sqrt{a}$, then for φ we have in this case

$$\bar{\varphi} = -(\varepsilon^2/p^2)\,e^{p\eta} K_0(p\eta\sqrt{a}). \tag{17}$$

This expression can be represented in the form of a product of two functions of p which are Laplacian forms of known functions. Utilizing the convolution theorem, we find that

$$\varphi(\xi, \eta) = -\varepsilon^2 \int_0^{\xi-(\sqrt{a}-1)\eta} [\xi - \tau - (\sqrt{a}-1)\eta] \frac{d\tau}{\sqrt{\tau^2 + 2\sqrt{a}\eta\tau}}. \tag{18}$$

Calculating the integral in (18), we obtain

$$\varphi(\xi, \eta) = -\varepsilon^2\sqrt{a}\eta\left\{\left[\left(\frac{\xi}{\sqrt{a}\eta}\right) + \frac{1}{\sqrt{a}}\right] \operatorname{Arch}\left[\left(1 + \left(\frac{\xi}{\sqrt{a}\eta}\right) - \right.\right.\right.$$

$$\left.\left.\left. - \left(\frac{\sqrt{a}-1}{\sqrt{a}}\right)\right]-\sqrt{\left[\left(\frac{\xi}{\sqrt{a}\eta}\right) + \frac{1}{\sqrt{a}}\right]^2 - 1}\right\} \tag{19}$$

and, returning to the dimensional variables, we finally arrive at the customary expression for the potential of conical flow

$$\varphi(x, r) = -\varepsilon^2 x U_\infty \{\operatorname{Arch}(x/\lambda_e r) - \sqrt{1 - (\lambda_e r/x)^2}\}. \tag{20}$$

In (20) the Mach number is determined according to the equilibrium velocity of sound.

This result is not unexpected, since the region $x/K \gg 1$ and $r/K \gg 1$ is a region of equilibrium flow.

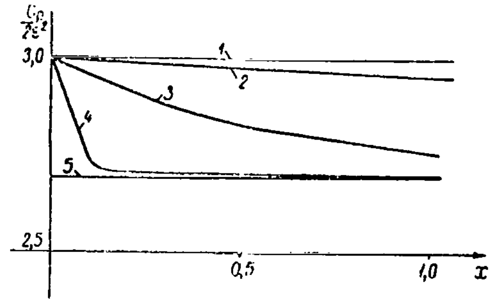

Figure 2
Dependence of C_p on Distance for a Cone with Half-Angle at Apex $\varepsilon = 5°$, $M_j = 1.2$ and $\alpha = 2$:

1 - $K = \infty$; 2 - $K = 10$; 3 - $K = 1$; 4 - $K = 0.1$; 5 - $K = 0$

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